ALM - Mean-Variance Analysis

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Course Program

- Basic interest rate theory
- Interest rate risk management
- Stochastic term structure models
- Risk measurement
- Reinsurance and insurance-linked securities
- `Mean-variance analysis for ALM`

Introduction

- The goal of this sectin is to show how to hedge a stochastic liability using correlated assets.
- We will also see how to simulate the stochastic development of correlated assets and liabilities.
- Make sure you understand the `matrice operations` and the simulations.

Contents organization

- Optimum asset allocation for one period
  - Minimum risk portfolio
  - Optimal portfolio of risky assets
  - Optimal portfolio with a risk-free asset
- Optimum asset allocation to fund a stochastic liability
  - Minimum risk portfolio
  - Optimal portfolio of risky assets
  - Optimal portfolio with a risk-free asset
- Discussion of the mean-variance framework.

MVA: Optimum asset allocation - I

- Assume that you can invest an amount of $W(0)$ in n different assets numbered $1, \cdots, n$.
- The market value of the assets now is $A_1(0), \cdots, A_n(0)$.
- You will revalue your assets at time $t > 0$.
- The market value of the assets will be $A_1(t), \cdots, A_n(t)$. This value must include the value of coupons or dividends paid during the period $(0, t]$.
- To all but the insiders, the outcome of $A_1(t), \cdots, A_n(t)$ looks random.

MVA: Optimum asset allocation - II

- Define the `return` of asset i by

$$R_i(t) = \frac{A_i(t) - A_i(0)}{A_i(t)}.$$

- If you invest $w_i W(0)$ in asset i at time 0, your wealth at time t will be

$$W(t) = W(0) \sum_{i=1}^{n} w_i \left(1 + R_i(t)\right) = W(0)(1 + \mathbf{w}' \mathbf{R}(t)),$$

MVA: Optimum asset allocation - III

- Your aggretate return over the period will be

$$R_{\mathbf{w}}(t) = \mathbf{w}^{'}\mathbf{R}(t).$$

- The asset allocation problem is to find a vector $\mathbf{w} = (w_1, \cdots, w_n)$ with $w_1 + \cdots + w_n = 1$ that provides an adequate expected return with as little as possible uncertainty.
- You decide to measure the uncertainty of $\mathbf{w}^{'}\mathbf{R}(t)$ by its variance.

MVA: Optimum asset allocation - IV

- In mathematical terms, the `asset allocation problem` then becomes "*minimise* $\mathrm{Var}\left(\mathbf{w}'\mathbf{R}(t)\right)$, *subject to certain constraints*".
- Conceivable investment constraints could be
  - `No constraints` at all, i.e. outright minimisation of the variance;
  - An `adequate expected return` r, i.e. $\mathrm{E}(\mathbf{w}'\mathbf{R}(t)) = r$;
  - `Exposure limits`, e.g. $w_{\min} \leq w \leq w_{\max}$.

MVA: Optimum asset allocation - V

We drop t for now, as we are considering only one period.

- Assume that the `return vector` $\mathbf{R} = (R_1, \cdots, R_n)'$ is random with a known mean vector

$$\mu = \mathrm{E}(\mathbf{R}) = (\mu_1, \cdots, \mu_n)'$$

- and a known `covariance matrix`

$$\Sigma = \mathrm{Cov}(\mathbf{R}) = \begin{pmatrix} \sigma_1^2 & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \vdots & \ddots & \vdots \\ \rho_{n1}\sigma_n\sigma_1 & \cdots & \sigma_n^2 \end{pmatrix}$$

MVA: Optimum asset allocation - VI

- We assume that there are `only risky assets`: there exists neither an asset i nor a linear combination (portfolio) of assets with a secure return.
- In that case the covariance matrix Σ is `invertible` and `positive definite`.
- A portfolio characterized by the allocation vector $\mathbf{w}$ has expected return and variance as follows:

$$\mathrm{E}(\mathbf{w}'\mathbf{R}) = \mathbf{w}'\mu, \quad \mathrm{Var}(\mathbf{w}'\mathbf{R}) = \mathbf{w}'\Sigma\mathbf{w}.$$

Asset behaviour example - assumptions

Assumptions

Asset class	mu	SD(return)
GBP	5.25 %	7.46 %
USD	5.50 %	5.85 %
CHF	2.00 %	4.13 %

Sigma	GBP	USD	CHF
GBP	5.57E-03	1.65E-03	4.33E-04
USD	1.65E-03	3.43E-03	7.07E-04
CHF	4.33E-04	7.07E-04	1.71E-03

EUR (Risk free rate mu_0)	1.00 %
Required expected return r	4.50 %
Initial funding ratio F	100.0 %

Sigma^(-1)	GBP	USD	CHF
GBP	2.10E+02	-9.82E+01	-1.26E+01
USD	-9.82E+01	3.65E+02	-1.26E+02
CHF	-1.26E+01	-1.26E+02	6.41E+02

https://www.global-rates.com/en/interest-rates/central-banks/central-banks.aspx

Minimum Variance Portfolio

- A very `variance-averse` investor could pose the asset allocation problem

$$\min_{\mathbf{w}} \mathbf{w}'\Sigma\mathbf{w}, \text{ subject to (only) } \mathbf{w}'\mathbf{1} = 1.$$

- Using Lagrange minimization, the optimal portfolio can be shown to be

$$\mathbf{w}_{\min} = (\mathbf{1}'\Sigma\mathbf{1})^{-1}\Sigma^{-1}\mathbf{1}$$

- Its return has expected value and variance as follows:

$$\mu'\mathbf{w}_{\min} = (\mathbf{1}'\Sigma\mathbf{1})^{-1}\mu'\Sigma^{-1}\mathbf{1}; \quad \mathbf{w}'_{\min}\Sigma\mathbf{w}_{\min} = (\mathbf{1}'\Sigma^{-1}\mathbf{1})^{-1}$$

Outline of proof

- The Lagrangian can be written as

$$\mathrm{L}(\mathbf{w}, \lambda) = \frac{1}{2}\mathbf{w}'\Sigma\mathbf{w} - \lambda(\mathbf{w}'\mathbf{1} - 1).$$

- To determine $\mathbf{w}_{\min}$ we solve the linear equations

$$\frac{\partial}{\partial \mathbf{w}}\mathrm{L}(\mathbf{w}, \lambda) = \mathbf{w}'\Sigma - \lambda\mathbf{1}' = 0',$$

$$\frac{\partial}{\partial \lambda}\mathrm{L}(\mathbf{w}, \lambda) = \mathbf{w}'\mathbf{1} - 1 = 0.$$

Example Min. Variance Portfolio

```
> w_min
          [,1]
[1,] 0.1332757
[2,] 0.1894328
[3,] 0.6772915
> # Expected asset return (3,10%)
> exp.returns.min <- mu %*% w_min
> exp.returns.min
           [,1]
[1,] 0.03096161
> # Variance and standard deviation of asset return
> var.returns.min <- t(w_min) %*% Sigma %*% w_min
> var.returns.min
            [,1]
[1,] 0.001347519
> sd.returns.min <- sqrt(var.returns.min)
> sd.returns.min
           [,1]
[1,] 0.03670857
```

Optimal portfolio of risky assets - I

- A more demanding investor could pose the asset allocation problem

$$\min_{\mathbf{w}} \mathbf{w}'\Sigma\mathbf{w}, \text{ subject to } \mathbf{w}'\mu = \text{r and (of course) } \mathbf{w}'\mathbf{1} = 1,$$

where r is the `expected return that an allocation must provide` in order to be a candidate.

- The optimal portfolio $\mathbf{w}_r$ is now a `linear combination` of the minimum variance portfolio $\mathbf{w}_{\min}$ and one “reference” risky portfolio $\mathbf{w}_{\text{ref}}$:

$$\mathbf{w}_r = (1 - v)\mathbf{w}_{\min} + v\mathbf{w}_{\text{ref}}$$

Optimal portfolio of risky assets - II

- The `reference risky portfolio` is

$$\mathbf{w}_{\text{ref}} = \left(\mathbf{1}'\Sigma^{-1}\mu\right)^{-1}\Sigma^{-1}\mu$$

  - or, in special cases, $\mathbf{w}_{\text{ref}} = \mathbf{w}_{\text{min}} + \Sigma^{-1}\mu$.

- The `weight` of the risky portfolio in the optimal portfolio is

$$v = v(r) = \frac{r - \mu'\mathbf{w}_{\text{min}}}{\mu'\mathbf{w}_{\mathbf{ref}} - \mu'\mathbf{w}_{\text{min}}}.$$

- The more return you ask for, the more risk you must accept.

Outline of proof - I

- The Lagrangian can be written as

$$L(\mathbf{w}, \lambda_1, \lambda_2) = \frac{1}{2}\mathbf{w}'\Sigma\mathbf{w} - \lambda_1(\mathbf{w}'\mathbf{1} - 1) - \lambda_2(\mathbf{w}'\mu - r).$$

- To determine $\mathbf{w}_r$ we solve the linear equations

1. $\frac{\partial}{\partial \mathbf{w}} L(\mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}'\Sigma - \lambda_1\mathbf{1}' - \lambda_2\mu' = \mathbf{0}'$
2. $\frac{\partial}{\partial \lambda_1} L(\mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}'\mathbf{1} - 1 = 0$
3. $\frac{\partial}{\partial \lambda_2} L(\mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}'\mu - r = 0$

Outline of proof - II

- Using 1 we find that the solution $\mathbf{w}$ is of the form

$$\mathbf{w} = \lambda_1\Sigma^{-1}\mathbf{1} + \lambda_2\Sigma^{-1}\mu = \lambda_1(\mathbf{1}'\Sigma^{-1}\mathbf{1})\mathbf{w}_{\text{min}} + \lambda_2\Sigma^{-1}\mu.$$

- Inserting this into 2 we find

$$\lambda_1(\mathbf{1}'\Sigma^{-1}\mathbf{1}) = 1 - \lambda_2(\mathbf{1}\Sigma^{-1}\mu).$$

- If $\mathbf{1}\Sigma^{-1}\mu \neq 0$, we can write

$$\mathbf{w} = (1 - \nu)\mathbf{w}_{\text{min}} + \nu\mathbf{w}_{\text{ref}},$$

Outline of proof - III

with a reference portfolio that is

$$\mathbf{w} = (\mathbf{1}'\Sigma^{-1}\mu)^{-1}\Sigma^{-1}\mu.$$

- If $\mathbf{1}\Sigma^{-1}\mu = 0,$ we can still write

$$\mathbf{w} = (1 - \nu)\mathbf{w}_{\min} + \nu\mathbf{w}_{\mathrm{ref}},$$

but the reference portfolio becomes (proof as an exercise)

$$\mathbf{w}_{\mathrm{ref}} = \mathbf{w}_{\min} + \Sigma^{-1}\mu.$$

- Finally, we solve 3 to determine the weight of the reference portfolio:

$$v = v(r) = \frac{r - \mu' \mathbf{w}_{\min}}{\mu' \mathbf{w}_{\text{ref}} - \mu' \mathbf{w}_{\min}}$$

Example Reference Risky Portfolio

```
> w_ref
            [,1]
[1,] 0.2331436
[2,] 0.5393612
[3,] 0.2274952
> # Expected asset return (3,10%)
> exp.returns.ref <- mu %*% w_ref
> exp.returns.ref
            [,1]
[1,] 0.04645481
> # Variance and standard deviation of asset return
> var.returns.ref <- t(w_min) %*% sigma %*% w_ref
> var.returns.ref
             [,1]
[1,] 0.001347519
> sd.returns.ref <- sqrt(var.returns.ref)
> sd.returns.ref
            [,1]
[1,] 0.03670857
```

Example - Optimal portfolio of risky assets with return requirement

```
> # Optimal portfolio of risky assets with return requirement
> w_r <- (1 - nu_r) * w_min + (nu_r * w_ref)
> w_r
         [,1]
[1,] 0.223766
[2,] 0.506503
[3,] 0.269731
> # Expected asset return
> exp.return.r <- mu %*% w_r
> exp.return.r
      [,1]
[1,] 0.045
> # Variance and standard deviation of asset return
> var.returns.r <- t(w_r) %*% Sigma %*% w_r
> var.returns.r
           [,1]
[1,] 0.00190113
> sd.returns.r <- sqrt(var.returns.r)
> sd.returns.r
           [,1]
[1,] 0.04360195
```

The efficient frontier of risky assets

- Any required (expected) return r can be generated by the formula

$$\mathbf{w}_r = (1 - v(r))\mathbf{w}_{\min} + v(r)\mathbf{w}_{\text{ref}},$$

- and the variance of the return will be the least possible:

$$\begin{aligned}
\sigma^2(r) &= \mathbf{Var}(\mathbf{w}_r^{'}\mathbf{R}) \\
&= (1 - v(r))^2\mathbf{w}_{\min}^{'}\Sigma\mathbf{w}_{\min} + v^2(r)\mathbf{w}_{\text{ref}}^{'}\Sigma\mathbf{w}_{\text{ref}} \\
&+ 2(1 - v(r))v(r)\mathbf{w}_{\min}^{'}\Sigma\mathbf{w}_{\text{ref}}.
\end{aligned}$$

- The `efficient frontier of risky assets` is the curve

$$\{(\sigma(r), r) : r \geq \mu^{'}\mathbf{w}_{\min}\}$$

Optimal portfolio with a risk-free asset - I

- Assume now that in addition to the n risky assets, you can invest in a `risk-free asset` $(i = 0)$ that provides a `secure return` of $R_0 = \mu_0$.
- Your asset allocation problem now becomes

$$\min_{w_0, \mathbf{w}} \mathbf{w}'\Sigma\mathbf{w}, \quad \text{subject to} \quad w_0\mu_0 + \mathbf{w}'\mu = r \quad \text{and} \quad w_0 + \mathbf{w}'\mathbf{1} = 1,$$

- where r is the expected return that an allocation must provide in order to be a candidate,
- and w_0 is the proportion of your wealth to be invested risk-free.

Optimal portfolio with a risk-free asset - II

- In this case, the optimal portfolio is a `combination` of
  - a risk-free investment of w_0 and
  - investment of the remaining $1 - w_0$ in a tangency portfolio $\mathbf{w}_{\tan}$.
- The relevant parameters are

$$\mathbf{w}_{\tan} = \mathbf{w}_{\tan}(\mu_0) = \left(\mathbf{1}'\Sigma^{-1}(\mu - \mu_0\mathbf{1})\right)^{-1}\Sigma^{-1}(\mu - \mu_0\mathbf{1})$$

$$r - \mu_0$$

Outline of proof I

- The Lagrangian can be written as

$$L(\mathbf{w}, \lambda_1, \lambda_2) = \frac{1}{2}\mathbf{w}'\Sigma\mathbf{w} - \lambda_1(w_0 + \mathbf{w}'\mathbf{1} - 1) - \lambda_2(w_0\mu_0 + \mathbf{w}'\mu - r).$$

- To determine the optimal $(w_0, \mathbf{w})$ we solve the linear equations

4. $\frac{\partial}{\partial \mathbf{w}} L(w_0, \mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}'\Sigma - \lambda_1\mathbf{1}' - \lambda_2\mu' = \mathbf{0}'$

5. $\frac{\partial}{\partial \mathbf{w}_0} L(w_0, \mathbf{w}, \lambda_1, \lambda_2) = -\lambda_1 - \lambda_2\mu_0 = 0$

6. $\frac{\partial}{\partial \lambda_1} L(w_0, \mathbf{w}, \lambda_1, \lambda_2) = w_0 + \mathbf{w}'\mathbf{1} - 1 = 0$

7. $\frac{\partial}{\partial \lambda_2} L(w_0, \mathbf{w}, \lambda_1, \lambda_2) = w_0\mu_0 + \mathbf{w}'\mu - r = 0$

Outline of proof II

- Using `4.` we find that the solution $\mathbf{w}$ is of the form

$$\mathbf{w} = \lambda_1 \Sigma^{-1} \mathbf{1} + \lambda_2 \Sigma^{-1} \mu.$$

- Using `5.` we find that $\lambda_1 = -\lambda_2 \mu_0$, so that

$$\mathbf{w} = \lambda_2 \Sigma^{-}1(\mu - \mu_0 \mathbf{1})$$

- Using `6.` we find $\lambda_2 = \frac{1 - w_0}{\mathbf{1}' \Sigma^{-1} (\mu - \mu_0 \mathbf{1})}$, so that $\mathbf{w} = (1 - w_0)\mathbf{w}_{\mathbf{tan}}.$

Outline of proof III

- Finally, `7.` gives us

$$1 - w_0 = 1 - w_0(r) = \frac{r - \mu_0}{\mu^{'}\mathbf{w}\tan - \mu_0}.$$

- Note that the tangency portfolio is a function of the available risk-free return.
- The variance of the overall return is

$$\sigma^2(r) = \mathrm{Var}(w_0\mu_0 + (1 - w_0)\mathbf{w}^{'}_{\tan}\mathrm{R}) = (1 - w_0)^2\mathbf{w}^{'}_{\tan}\Sigma\mathbf{w}_{\tan}.$$

Example - Tangency Portfolio

```
> # Tangency portfolio
> w_tan <- as.numeric((ones.row %*% Sigma.inv %*% (mu.col - mu_0 * ones.col))^(-1)) *
+ Sigma.inv %*% (mu.col - mu_0 * ones.col)
> w_tan
           [,1]
[1,] 0.28078682
[2,] 0.70629904
[3,] 0.01291414
> w_tan
           [,1]
[1,] 0.28078682
[2,] 0.70629904
[3,] 0.01291414
> # Variance and standard deviation of asset return
> var.returns.tan <- t(w_tan) %*% Sigma %*% w_tan
> var.returns.tan
            [,1]
[1,] 0.002818647
> sd.returns.tan <- sqrt(var.returns.tan)
> sd.returns.tan
           [,1]
[1,] 0.05309093
```

Example

- Optimal Portfolio of risky assets and a risk-free asset with return requirement

```
> # Weight of the tangency portfolio
> weight.tan <- (r - mu_0) / as.numeric(mu.row %*% w_tan - mu_0)
> weight.tan
[1] 0.7982477
> # Weight of the risk free asset
> w_0 <- 1 - weight.tan
> w_0
[1] 0.2017523
> # Variance and standard deviation of the overall return
> var.returns.overall <- (1 - w_0) * var.returns.tan
> var.returns.overall
            [,1]
[1,] 0.002249978
> sd.returns.overall <- sqrt(var.returns.overall)
> sd.returns.overall
           [,1]
[1,] 0.04743394
```

Optimum asset allocation with hedge

- Let us briefly reintroduce the time parameter $t > 0$.
- Assume that the assets must support a `stochastic liability`.
- The value of the liability at time 0 is $\mathrm{L}(0)$, and the time t it will be $\mathrm{L}(t)$.
- The surplus at time 0 is $\mathrm{S}(0) = \mathrm{W}(0) - \mathrm{L}(0)$. At time t it will be $\mathrm{S}(t) = \mathrm{W}(t) - \mathrm{L}(t)$.
- The funding ratio at time 0 is $\mathrm{F}(0) = \dfrac{\mathrm{W}(0)}{\mathrm{L}(0)}$.

Optimum asset allocation with hedge

- Sharpe & Tint (1990) define the `surplus return` as

$$\frac{S(t) - S(0)}{W(0)} = \left(\frac{W(t) - W(0)}{W(0)}\right) - \frac{L(0)}{W(0)}\left(\frac{L(t) - L(0)}{L(0)}\right)$$

$$= R_W(t) - \frac{R_L}{F(0)}.$$

- Here we have defined:
  - $R_W(t) = \frac{W(t) - W(0)}{W(0)}$ as asset return
  - $R_L(t) = \frac{L(t) - L(0)}{L(0)}$ as liability growth.

Assumptions - I

- Let us assume that there are n investible assets with a random return characterized by its mean vector and covariance matrix:

$$\mathbf{R} \sim [\mu(t), \Sigma(t)]$$

- We now make the `additional assumption` that liability growth is random and correlated with asset returns:

$$\begin{aligned} \mathrm{E}(\mathrm{R}_{\mathrm{L}}(t)) &= \mu_{\mathrm{L}}(t) \\ \mathrm{Var}(\mathrm{R}_{\mathrm{L}}(t)) &= \sigma^2_{\mathrm{L}}(t) \\ \mathrm{Cov}(\mathrm{R}_i(t), \mathrm{R}_{\mathrm{L}}(t)) &= \gamma_{i,\mathrm{L}}(t) = \rho_{i,\mathrm{L}}(t)\sigma_i(t)\sigma_{\mathrm{L}}(t) \end{aligned}$$

Assumptions - II

- Denote the vector of covariances by

$$\gamma(t) = (\gamma_{1,\mathrm{L}}(t), \cdots, \gamma_{n,\mathrm{L}}(t))'$$

  and assume you know (have estimated) $\mu_{\mathrm{L}}(t)$, $\sigma_{\mathrm{L}}^2(t)$ and $\gamma(t)$.

- Let us now find `optimal asset allocations to fund a stochastic liability`.

Optimum asset allocation with hedge

- With an arbitrary asset allocation vector **w**, the random surplus return is

$$R_S(t) = \mathbf{w}'\mathbf{R}(t) - \frac{R_L(t)}{F(0)} = \mathbf{w}'\mathbf{R} - \frac{R_L}{F} = R_S$$

- It is easy to verify that

$$E(R_S) = \mathbf{w}'\mu - \frac{\mu_L}{F}$$

$$\mathrm{Var}(R_S) = \mathbf{w}'\Sigma\mathbf{w} + \frac{\sigma_L^2}{F^2} - 2\frac{\mathbf{w}'\gamma}{F}$$

- Let us minimize the variance, subject to the constraints.

Minimum Variance Portfolio - Hedge

- If your only aim is to `minimize variance`, you would solve:

$$\min_{\mathbf{w}} \left(\mathbf{w}'\Sigma\mathbf{w} + \frac{\sigma_L^2}{F^2} - 2\frac{\mathbf{w}'\gamma}{F} \right) \text{ subject to } \mathbf{w}'\mathbf{1} = 1$$

- Using Lagrange minimization, the optimal portfolio can be shown to be

$$\mathbf{w}_{\min}(F, \gamma) = (1 - \nu)\mathbf{w}_{\min} + \nu\mathbf{w}_{\gamma}$$

where $\mathbf{w}_{\min}$ is the unconditional minimum variance portfolio and $\mathbf{w}_{\gamma}$ is the `liability hedge portfolio`.

Liability Hedge Portfolio

- The liability hedge portfolio is

$$\mathbf{w}_{\gamma} = (\mathbf{1}'\Sigma^{-1}\gamma)^{-1}\Sigma^{-1}\gamma.$$

- The weight of the liability hedge portfolio in the optimal portfolio is

$$v = v(\mathrm{F}, \gamma) = \frac{1}{\mathrm{F}}\mathbf{1}'\Sigma^{-1}\gamma.$$

- In the case where $\mathbf{1}'\Sigma^{-1}\gamma = 0,$ we can write $\mathbf{w}_{\gamma} = \mathbf{w}_{\min} + \Sigma^{-1}\gamma$ and $v = 1.$

Outline of proof - I

- The Lagrangian can be written as *[Math Processing Error]*
- To determine $\mathbf{w}$ we solve the linear equations *[Math Processing Error]*

Outline of proof - II

- The first equation gives

$$\mathbf{w} = \lambda \Sigma^{-1} \mathbf{1} + \frac{1}{F} \Sigma^{-1} \gamma'$$

- and the second equation gives

$$\lambda = (\mathbf{1}`\Sigma^{-1}\mathbf{1})-1 \left(1 - \frac{1}{F} \mathbf{1}' \Sigma^{-1} \gamma \right).$$

- Proceed from there.

Comments

- If asset-liability `covariance is small` relative to the asset variability, then v will be small and the optimal portfolio will be close to $\mathbf{w}_{\text{min}}$.
- In particular, if there is no asset-liability covariance then the optimal portfolio is just $\mathbf{w}_{\text{min}}$.
- The weight given to the liability hedge portfolio is a `decreasing` function of the initial funding ratio.

Example - Liability hedge portfolio

```
> # First, we compute the covariances between our assets and our liability
> gamma <- rho * sd.assets * sd.liab
> gamma
[1] 0.001398924 0.002195575 0.002323767
> gamma.col <- matrix(gamma, ncol = 1)
>
>
> # Liability hedge portfolio
>
> w_gamma <- (as.numeric(ones.row %*% Sigma.inv %*% gamma) ^ -1) * Sigma.inv %*% gamma
> w_gamma
           [,1]
[1,] 0.03002953
[2,] 0.22952535
[3,] 0.74044512
>
> # Expected asset return (homework - 2.90%)
>
>
> # Variance of asset return (homework - 0.0014)
>
>
> # SD of asset return (homework - 3.75%)
```

Example - Minimum variance portfolio with hedge

```
> w_min_F_gamma
            [,1]
[1,] -0.03346542
[2,]  0.25418172
[3,]  0.77928369
> # Expected asset return
> mu.row %*% w_min_F_gamma
           [,1]
[1,] 0.02780873
> # Expected surplus return
> mu.surplus <- as.numeric(mu.row %*% w_min_F_gamma - mu.liab / F_0)
> mu.surplus
[1] -0.01719127
> # Variance of asset return
> t(w_min_F_gamma) %*% Sigma %*% w_min_F_gamma
            [,1]
[1,] 0.001493428
> # Variance of surplus return
> var.surplus <- t(w_min_F_gamma) %*% Sigma %*% w_min_F_gamma +
+   (sd.liab / F_0)^2 - 2*(t(w_min_F_gamma) %*% gamma.col) / F_0
> var.surplus
            [,1]
[1,] 0.002474161
> # Standard deviation of asset return
> sqrt(t(w_min_F_gamma) %*% Sigma %*% w_min_F_gamma)
          [,1]
[1,] 0.0386449
> # Standard deviation of surplus return
> sqrt(var.surplus)
           [,1]
[1,] 0.04974094
```

Optimal portfolio of risky assets with hedge I

- If one wants to beat instead of mee the expected return of the liability hedge portfolio, one could solve:

$$\min_{\mathbf{w}} \left(\mathbf{w}^{\prime} \Sigma \mathbf{w} + \frac{\sigma_{\mathrm{L}}^2}{\mathrm{F}^2} - 2 \frac{\mathbf{w}^{\prime} \gamma}{\mathrm{F}} \right) \text{ subject to } \mathbf{w}^{\prime} \mu = r \text{ and } \mathbf{w}^{\prime} \mathbf{1} = 1$$

  where r is the expected return required.

- This only makes sense if $r \geq \mu^{\prime} \mathbf{w}_{\min}(\mathrm{F}, \gamma)$.

Portfolio of risky assets with hedge II

- The optimal portfolio can be written as:

$$
\begin{aligned}
\mathbf{w}_r(\mathrm{F}, \gamma) &= (1 - \nu - \omega)\mathbf{w}_{\min} + \omega \mathbf{w}_{\mathrm{ref}} + \nu \mathbf{w}_\gamma \\
&= \mathbf{w}_{\min}(\mathrm{F}, \gamma) + \omega(\mathbf{w}_{\mathrm{ref}} - \mathbf{w}_{\min})
\end{aligned}
$$

  - $\mathbf{w}_{\min}$ is the unconditional minimum variance allocation,
  - $\mathbf{w}_{\mathrm{ref}}$ is the reference risky portfolio without a risk-free asset,
  - $\mathbf{w}_\gamma$ is the `liability hedge` portfolio,
  - $\mathbf{w}_{\min}(\mathrm{F}, \gamma)$ is the `minimum surplus variance` allocation.
- The weighting parameters are

Outline of proof

- The Lagrangian can be written as

$$L(\mathbf{w}, \lambda_1, \lambda_2) = \frac{1}{2}\left(\mathbf{w}'\Sigma\mathbf{w} + \frac{\sigma_L^2}{F^2} - 2\frac{\mathbf{w}'\gamma}{F}\right) - \lambda_1(\mathbf{w}'\mathbf{1} - 1) - \lambda_2(\mathbf{w}'\mu - r).$$

- To determine $\mathbf{w}$ we solve the linear equations

$$\frac{\partial}{\partial \mathbf{w}} L(\mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}'\Sigma - \frac{1}{F}\gamma' - \lambda_1\mathbf{1}' - \lambda_2\mu' = \mathbf{0}',$$

$$\frac{\partial}{\partial \lambda_1} L(\mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}'\mathbf{1} - 1 = 0$$

$$\frac{\partial}{\partial \lambda_2} L(\mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}'\mu - r = 0$$

and so on…

Example

- Optimal portfolio of risky assets to fund a stochastic liability

```
> w_min_F_gamma
            [,1]
[1,] -0.03346542
[2,]  0.25418172
[3,]  0.77928369
> # Weight omega
> omega <- (r - mu.row %*% w_min_F_gamma) / (mu.row %*% w_ref - mu.row %*% w_min)
> omega <- as.numeric(omega)
> omega
[1] 1.109601
> # Apply the formula
> w_r_F_gamma <- w_min_F_gamma + omega * (w_ref - w_min)
> w_r_F_gamma
           [,1]
[1,] 0.07734805
[2,] 0.64246252
[3,] 0.28018942
```

Portfolio of risky assets with a risk-free asset with hedge - I

- Finally, let us develop the case where the investor has access to a `risk-free asset` with return μ_0, i.e.,

$$\min_{\mathbf{w}} \left(\mathbf{w}'\Sigma\mathbf{w} + \frac{\sigma_L^2}{F^2} - 2\frac{\mathbf{w}'\gamma}{F} \right) \text{ subject to } w_0\mu_0 + \mathbf{w}'\mu = r$$

$$\text{and } \mathbf{w}_0 + \mathbf{w}'\mathbf{1} = 1$$

- The parameter w_0 denotes the proportion of assets invested

Portfolio of risky assets with a risk-free asset with hedge - II

- The optimal portfolio consists of
  - a risk-free investment of w_0,
  - investment of $1 - w_0 - v$ in portfolio $\mathbf{w}_{\mathbf{tan}}$,
  - investment of v in the liability hedge portfolio $\mathbf{w}_\gamma$.
- The weights are

$$v = \frac{1}{F}\mathbf{1}'\Sigma^{-1}\gamma \text{ and } 1 - w_0 = \frac{r - v\mu'(\mathbf{w}_\gamma - \mathbf{w}_{\text{tan}}) - \mu - 0}{\mu'\mathbf{w}\text{tan} - \mu_0}$$

- We omit the proof.

Example

- Optimal portfolio of risky assets with a risk-free asset to fund a stochastic liability

```
> w_min_F_gamma
            [,1]
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[2,]  0.25418172
[3,]  0.77928369
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           [,1]
[1,] 0.07734805
[2,] 0.64246252
[3,] 0.28018942
```

Discussion of mean-variance framework - I

- Using the mean-variance framework provides insight into the effect of `correlation` between asset classes, and between asset classes and liabilities.
- Estimating the covariances is easy in principle. Having to rely on estimated covariances in allocating your portfolio may be more problematic. It requires great confidence in the estimates.
- The mean-variance framework may return allocations that are `not feasible`, because they are outside the investment mandate. There exists software to minimize with arbitrary

Discussion of mean-variance framework - II

- Even if your asset allocation is subject to constraints, you should calculate the `cost` of those constraints in terms of lost return or increased volatility, relative to what an unconstrained allocation could achieve.

- Given the framework (`means and covariances`), the method returns an optimal asset allocation. *Optimal* does not necessarily mean *very good* - it just means the best that could be achieved under the given assumptions.

- Asset returns are `not normally distributed`! However, relying on means and covariances does not imply that you subscribe to the normality assumption. It only means that you select two readily available distribution characteristics and ignore the rest.